

# Topological Strings on Grassmannian Calabi-Yau manifolds

Babak Haghighat<sup>a</sup> and Albrecht Klemm<sup>a</sup>

<sup>a</sup> *Physikalisches Institut, Universität Bonn,  
D-53115 Bonn, BRD*

## Abstract

We present solutions for the higher genus topological string amplitudes on Calabi-Yau-manifolds, which are realized as complete intersections in Grassmannians. We solve the B-model by direct integration of the holomorphic anomaly equations using a finite basis of modular invariant generators, the gap condition at the conifold and other local boundary conditions for the amplitudes. Regularity of the latter at certain points in the moduli space suggests a CFT description. The A-model amplitudes are evaluated using a mirror conjecture for Grassmannian Calabi-Yau by Batyrev, Ciocan-Fontanine, Kim and Van Straten. The integrality of the BPS states gives strong evidence for the conjecture.

February 21, 2008

---

<sup>a</sup>babak@th.physik.uni-bonn.de, aklemm@th.physik.uni-bonn.de

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Calabi-Yau complete intersections in Grassmannians</b>	<b>3</b>
2.1	Topological invariants of the manifolds . . . . .	3
2.2	Plücker embedding . . . . .	5
2.3	Mirror Construction . . . . .	6
<b>3</b>	<b>The BCOV anomaly equation</b>	<b>6</b>
3.1	Special geometry and the topological string . . . . .	7
3.2	General solutions of the BCOV anomaly equation . . . . .	9
3.3	Topological limit and Gromov-Witten potentials . . . . .	11
<b>4</b>	<b>The Grassmannian Calabi-Yau <math>(\mathbb{G}(2, 5) \parallel 1, 1, 3)_{-150}^1</math></b>	<b>12</b>
4.1	Picard-Fuchs differential equations and the structure of the moduli space	12
4.2	$g = 0$ and $g = 1$ Gopakumar-Vafa invariants . . . . .	13
4.3	Higher genus GV-Invariants . . . . .	14
4.3.1	Recursive Solution of the BCOV equation . . . . .	14
4.4	Holomorphic ambiguity and boundary conditions . . . . .	17
4.4.1	Expansion around the conifold points . . . . .	17
4.4.2	Expansion around the orbifold point . . . . .	19
<b>5</b>	<b>Other Models</b>	<b>21</b>
5.1	$(\mathbb{G}(2, 5) \parallel 1, 2, 2)_{-120}^1$ . . . . .	21
5.2	$(\mathbb{G}(3, 6) \parallel 1^6)_{-96}^1$ . . . . .	22
5.3	$(\mathbb{G}(2, 6) \parallel 1, 1, 1, 1, 2)_{-116}^1$ . . . . .	23
5.4	$(\mathbb{G}(2, 7) \parallel 1^7)_{-98}^1$ . . . . .	24
<b>6</b>	<b>Conclusions</b>	<b>25</b>
<b>A</b>	<b>Chern classes and topological invariants</b>	<b>27</b>
<b>B</b>	<b>Tables of Gopakumar-Vafa invariants</b>	<b>27</b>
<b>C</b>	<b>5D Blackhole asymptotic</b>	<b>33</b>

# 1 Introduction

Mirror symmetry of Calabi-Yau manifolds has been understood to large extend for complete intersections or hypersurfaces in toric ambient space. However a huge and much less explored class of Calabi-Yau manifolds, with distinct low energy spectrum, can be realized in ambient spaces, which are defined by other homogeneous spaces like the Grassmannians  $\mathbb{G}(k, n) = U(n)/(U(k) \times U(n-k))$ . The topological properties of spaces defined by the complex actions of Lie groups are described in [1]. From the point of the 2-d linear  $\sigma$ -model description of the ambient space [2] the difference is that the former have  $U(1)^r$  gauge symmetries, while the latter have non-abelian  $\prod_k U(N_k)$  gauge symmetries. The proof that mathematicians [3] gave for the fact that the  $B$ -model calculation of the genus zero amplitude counts worldsheet instantons on the mirror manifold  $W$  relies on localisation w.r.t. the  $U(1)^r$  action and the construction of mirror pairs by reflexive polyhedra. It has not been extended to the non-abelian case, e.g. to Grassmannian Calabi-Yau. For higher genus amplitudes such proofs are not in general available even on normal toric ambient spaces, but there are some results on genus one amplitudes [4][5]. In this article we explore the physical mirror symmetry predictions in situations, where it is mathematical very difficult to prove along the lines described above, namely for the higher genus amplitudes on Grassmannian Calabi-Yau spaces. Nevertheless the physical integrality conditions on the BPS invariants, defined in [6][7] give strong consistency checks on our A-model mirror symmetry predictions on these manifolds.

For genus zero the first steps in the B-model analysis for Grassmannian Calabi-Yau spaces have been done in [8]. Since the usual construction of mirror pairs by reflexive polyhedra does apply only to toric Calabi-Yau, the strategy of the authors is to consider a conifold transition from a Grassmannian Calabi-Yau to a toric Calabi-Yau, apply Batyrevs mirror construction there and perform an inverse conifold transition back to a Grassmannian Calabi-Yau. This is reviewed in section 2.3. For technical reasons we chose the new one parameter models, for which the mirror geometry and in particular the Picard-Fuchs equations were found in [8]. We apply the methods developed in [9][10][11] to the B-model. Notably the structure of the holomorphic and an-holomorphic modular expressions in the amplitudes analysed in [10] allows for a very effective recursive integration of the holomorphic anomaly equations. This structure can be related to the traditional theory of holomorphic and anholomorphic modular forms of subgroups of  $SL(2, \mathbb{Z})$  in the case of local mirror symmetry [12]. For the large moduli space of the Calabi-Yau this formalism can be extended at least formally to the global case [13]. The automorphic forms should be then associated to abelian varieties.

The direct integration of the holomorphic anomaly has to be supplemented with boundary conditions to provide the solutions. We find that the gap condition at the generic conifold divisor, where an  $S^3$  shrinks, found in [11] is present also in the Grassmannian Calabi-Yau spaces and provides most of the information. Other boundary information is provided by the regularity at CFT points in the moduli and places where lens spaces  $S^3/\mathbb{Z}_N$  shrink. The Picard-Fuchs equations of the one parameter Grassmannian Calabi-Yau spaces are considerably more involved than the ones for hypersurfaces and complete intersections in toric Calabi-Yau. While the latter have always three regular singular points in a  $\mathbb{P}^1$  compactification, the former have many regular singular points. One motivation for the investigation was to analyze the degeneration of the higher genus

amplitudes at these partly novel singularities and to see whether enough boundary conditions can be found to solve the theory completely. In all one parameter cases, one has been analyzed also in [14], we can use the methods described above to solve the model at least to genus 5 and in many cases higher.

## 2 Calabi-Yau complete intersections in Grassmannians

In this section we introduce the Calabi-Yau intersections in Grassmannian, calculate their topological data and review the mirror construction of [8].

### 2.1 Topological invariants of the manifolds

Compact Calabi-Yau manifolds  $M$  can be constructed by considering complete intersections in Kähler ambient spaces with positive Chern class. The first Chern class of the complete intersections is controlled by the adjunction formula and we can choose appropriate degrees of the complete intersection constraints so that  $c_1(TM) = 0$ . We will calculate the topological data of  $M$  by basic algebraic geometry. All necessary tools are reviewed in [15, 1].

We restrict to complete intersections in smooth Grassmannians. In this way one finds 5 complete intersections  $M$  with  $h^{1,1} = 1$ . The ambient space will be denoted as  $\mathbb{G}(k, n) = (U(k) \times U(n - k))$ , where  $U(n)$  are the unitary groups. For the complete intersection we use the notation

$$(\mathbb{G}(k, n) \| d_1, \dots, d_l)_{\chi}^{h^{1,1}}. \quad (2.1)$$

Here the degrees  $d_i$  of the Calabi-Yau intersection are given w.r.t. to the principal canonical bundle  $Q$  of the Grassmannian, see below. In addition we give the Euler number  $\chi$  as subscript and the Picard number  $h^{1,1}$  as superscript. Of course,  $h_{3,0} = 1$ ,  $h_{k,0} = 0$  for  $k = 1, 2$  and  $h^{2,1} = -\frac{\chi}{2} + h^{1,1}$ . Together with Poincaré and Hodge duality this fixes all Hodge numbers of  $M$ . All necessary topological data, which fix the topological type of  $M$ , are calculated below using Schubert calculus.

Let us first give a closed expression for the Chern classes of Grassmannians following Borel and Hirzebruch in [1]. Their method is based on an identification of Chern classes with elementary symmetric polynomials or combinations of them, which we will summarize here.

Let  $S\{x_1, \dots, x_l\}$  denote the set of elementary symmetric polynomials in the variables  $x_1, \dots, x_l$ . Then the integral homology  $H^*(\mathbb{G}(k, n), \mathbb{Z})$  of the Grassmannian can be identified with the quotient

$$S\{x_1, \dots, x_{n-k}\} \otimes S\{x_{n-k+1}, \dots, x_n\} / I, \quad (2.2)$$

where  $I$  is the ideal generated by the symmetric power series in  $x_1, \dots, x_n$  without constant term. Now, in this representation, the closed formula for the total Chern class reads

$$c(\mathbb{G}(k, n)) = \prod_{i=1}^{n-k} (1 - x_i)^n \prod_{1 \leq i \leq j \leq n-k} (1 - (x_i - x_j)^2)^{-1}. \quad (2.3)$$

Practically, in order to calculate the Chern classes, substitute each  $x_l$  by  $hx_l$  and make a series expansion in  $h$ . Then, the  $i$ 's Chern class is given by the coefficient of  $h^i$  which can be expressed in terms of elementary symmetric polynomials  $\sigma_r$ ,  $r \leq i$  in  $x_1, \dots, x_{n-k}$ . For example, we have

$$\begin{aligned} c_1(\mathbb{G}(k, n)) &= -n\sigma_1, \\ c_2(\mathbb{G}(k, n)) &= \left( \binom{n}{2} + n - k - 1 \right) \sigma_1^2 + k\sigma_2. \end{aligned} \quad (2.4)$$

The formula for the first Chern class shows that  $-\sigma_1$  is a positive generator of  $H^2(\mathbb{G}(k, n), \mathbb{Z})$ . Next, note that  $\sigma_r$  is (up to a possible sign) the  $r$ -th Chern class of the canonical principal  $U(n-k)$ -bundle  $Q$  over  $\mathbb{G}(k, n)$  and as such represents the class of a hyperplane section. We have  $\sigma_1 = -c_1(Q)$ ,  $\sigma_2 = c_2(Q)$ ,  $\sigma_3 = -c_3(Q)$ ,  $\dots$

Finally, we are ready to write down the total Chern class of Calabi-Yau complete intersections  $(\mathbb{G}(k, n) \| d_1, \dots, d_l)_\chi^{h^{1,1}}$ ,  $l = k(n-k) - 3$ ,  $d_1 + \dots + d_l = n$  :

$$c((\mathbb{G}(k, n) \| d_1, \dots, d_l)_\chi^{h^{1,1}}) = \frac{c(\mathbb{G}(k, n))}{(1 + d_1 c_1(Q)) \cdots (1 + d_l c_1(Q))}. \quad (2.5)$$

Denoting by  $H$  the hyperplane  $\sigma_1$ , the topological invariants  $\chi(M)$ ,  $c_2(M) \cdot H$ ,  $H^3$  can be expressed through intersection numbers of the Grassmannian  $\mathbb{G}(k, n)$ . As an example, we review the calculation of the Euler number. The Gauss-Bonnet formula gives  $\int_M c_3(M) = \chi$ . Now, using the adjunction formula, this integral can be expressed through an integral over the whole Grassmannian

$$\chi(M) = \int_M c_3(M) = \int_{\mathbb{G}(k, n)} c_3(M) \prod_{i=1}^l d_i H = \int_{\mathbb{G}(k, n)} c_3(M) \prod_{i=1}^l d_i c_1(Q). \quad (2.6)$$

Similarly, the other topological invariants are given by

$$c_2(M) \cdot H = \int_{\mathbb{G}(k, n)} c_2(M) c_1(Q) \prod_{i=1}^l d_i c_1(Q), \quad (2.7)$$

$$H^3 = \int_{\mathbb{G}(k, n)} c_1(M)^3 \prod_{i=1}^l d_i c_1(Q). \quad (2.8)$$

As all Chern classes of  $M$  are expressed through Chern classes of  $Q$ , which are Poincare dual to the Schubert cycles of the Grassmannian, all invariants can at the end be expressed through intersection numbers of Schubert cycles. These numbers can then be

calculated utilizing the Schubert calculus and Pieri's formula. Denoting by  $\sigma_a$  the special Schubert cycle given by the indices  $a = (a, 0, \dots, 0)$  and by  $\sigma_{\underline{b}}$  a general Schubert cycle with indices  $\underline{b} = (b_1, \dots, b_k)$ , Pieri's formula reads

$$\sigma_a \cdot \sigma_{\underline{b}} = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ \sum c_i = a + \sum b_i}} \sigma_{\underline{c}}. \quad (2.9)$$

Note that in the above formula the index  $c_1$  must always be greater or equal to  $b_1$ . For further details we refer to [15].

We have performed the above steps and list the result for our Calabi-Yau complete intersections in the Appendix.

## 2.2 Plücker embedding

In order to describe the mirror of the complete intersections in Grassmannians it is useful to have an embedding of the Grassmannian into the projective space. The Plücker map provides such an embedding. It simply sends a  $k$ -plane  $\Lambda = \mathbb{C}\{v_1, \dots, v_k\} \subset \mathbb{C}^n$  to the multivector  $v_1 \wedge \dots \wedge v_k$ .

Explicitly, in terms of the basis  $\{e_I = e_{i_1} \wedge \dots \wedge e_{i_k}\}_{\#I=k}$  for  $\wedge^k \mathbb{C}^n$ , this map is given by the data

$$p : \mathbb{G}(k, n) \rightarrow \mathbb{P}(\wedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}, \quad (2.10)$$

$$\Lambda \mapsto [\dots, |\Lambda_I|, \dots], \quad (2.11)$$

where the  $|\Lambda_I|$  are the determinants of all the  $k \times k$  minors of  $\Lambda_I$  of a matrix representative of  $\Lambda$ .

To describe this embedding algebraically we need to find a set of equations which cut out the Grassmannian in  $\mathbb{P}^{\binom{n}{k}-1}$ , i.e. which define conditions on a multivector  $\Lambda \in \wedge^k V$  to be of the form

$$\Lambda = v_1 \wedge \dots \wedge v_k. \quad (2.12)$$

Some calculations show that this is equivalent to demanding

$$(i(\Xi)\Lambda) \wedge \Lambda = 0, \quad (2.13)$$

for all  $\Xi \in \wedge^{k-1} V$ . Here, the map  $i(\Xi)\Lambda$  is defined by

$$\langle i(\Xi)\Lambda, v \rangle = \langle \Xi, \Lambda \wedge v \rangle \quad (2.14)$$

for all  $v \in V$ .

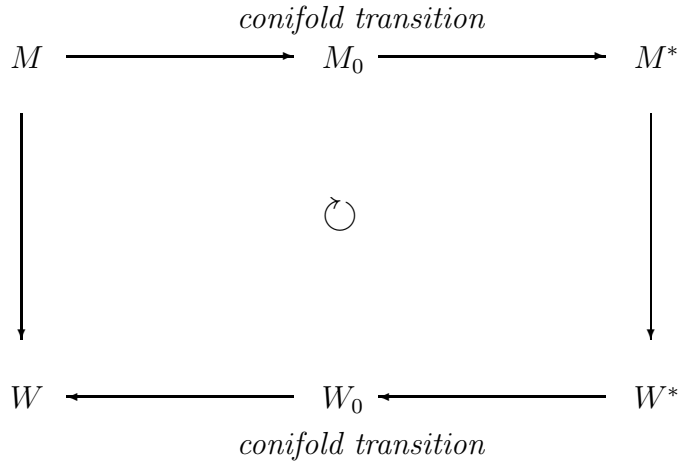
Now, a Calabi-Yau complete intersection is obtained by choosing hypersurfaces of appropriate total degree in  $\mathbb{P}^{\binom{n}{k}-1}$ , such that their intersection with  $\mathbb{G}(n, k)$  is a nonsingular Calabi-Yau space.

## 2.3 Mirror Construction

A mirror construction for the above type of Calabi-Yau spaces was given in [8]. Here, we will only sketch the method introduced there which is based on conifold transitions.

Let  $M$  be a Grassmannian Calabi-Yau described by the Grassmannian  $\mathbb{G}(k, n)$  and hyperplanes  $H_i$ . As was shown by Sturmfels [16] a flat deformation of  $\mathbb{G}(k, n)$  in its Pluecker embedding leads to a Gorenstein toric Fano variety  $P(k, n) \subset \mathbb{P}^{\binom{n}{k}-1}$ . Now, denote by  $M_0$  the intersection of  $P(k, n)$  with generic hypersurfaces  $H_i$ . This manifold has a locus of conifold singularities which come from the singularities of  $P(k, n)$ . Resolving these by restriction of a small toric resolution of singularities in  $P(k, n)$  one obtains a second Calabi-Yau  $M^*$ .  $M^*$  is a complete intersection in a toric manifold and as such its mirror construction is known. The remaining task is to find an appropriate specialization of the toric mirror  $W^*$  for  $M^*$  to a conifold  $W_0$  whose small resolution provides the mirror  $W$  of  $M$ . This task was performed in [8] for the manifolds we will be dealing with in this paper.

The above steps can be summarized in the following graph:



## 3 The BCOV anomaly equation

In this section, the general procedure for solving the BCOV anomaly equation is reviewed. The connection of the solutions to Gromov-Witten potentials is established which will allow us to extract Gopakumar-Vafa invariants from a series expansion of these potentials.

### 3.1 Special geometry and the topological string

Here we review how the deformation space of the topological B-model carries the structure of a special Kähler manifold which can be identified with the special Kähler geometry of local Calabi-Yau moduli spaces. As is discussed in [9], infinitesimal deformations of the topological B-model are parametrized by the chiral fields of charge  $(q, \bar{q}) = (1, 1)$ . These are the marginal fields which are spanned by a basis  $\phi_i$  for  $i = 1, \dots, n$ . In fact, the deformations span a complex manifold  $\mathcal{M}$  of dimension  $n$ . We are interested in the ring spanned by  $(\phi_0, \phi_i, \phi^i, \phi^0)$ , where  $\phi_0$  is the identity operator of charge  $(q, \bar{q}) = (0, 0)$ , and  $\phi^i$  are the charge  $(2, 2)$  fields and finally  $\phi^0$  is the top element in the chiral ring of charge  $(3, 3)$ . These fields satisfy the following identities with respect to the topological metric

$$\eta(\phi_i, \phi^j) = \langle \phi_i \phi^j \rangle_0 = \delta_i^j, \quad (3.1)$$

$$\eta(\phi_0, \phi^0) = \langle \phi_0 \phi^0 \rangle_0 = 1. \quad (3.2)$$

Here,  $\langle \cdot \rangle_0$  denotes the topological correlation function on the sphere. The ring structure is encoded in the so called Yukawa coupling, which is the three-point function on the sphere

$$C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle_0. \quad (3.3)$$

Using the operator state correspondence one can define

$$|i\rangle = \phi_i |0\rangle, \quad (3.4)$$

and the topological metric becomes

$$\eta(\phi_i, \phi^j) = \langle i | j \rangle = \delta_i^j. \quad (3.5)$$

Finally, one can define a hermitian metric using the worldsheet CPT operator  $\Theta$ ,

$$g_{i\bar{j}} = \langle \Theta j | i \rangle. \quad (3.6)$$

Now, moving around in the moduli space  $\mathcal{M}$ , the space of states generated by the chiral fields forms a holomorphic vector bundle  $V \rightarrow \mathcal{M}$ . It can be shown that its charge  $(0, 0)$  subspace forms a holomorphic line bundle  $\mathcal{L}$  over  $\mathcal{M}$  and that the charge  $(1, 1)$  subbundle corresponds to the line bundle  $\mathcal{L} \times T\mathcal{M}$ . The charge  $(2, 2)$  and  $(3, 3)$  subbundles respectively turn out to be duals of  $\mathcal{L} \times T\mathcal{M}$  and  $\mathcal{L}$ . These bundles are described through their covariant derivatives which will be given in the following. On  $\mathcal{M}$  one can define a metric, called Zamolodchikov metric,

$$G_{i\bar{j}} = \frac{g_{i\bar{j}}}{g_{0\bar{0}}}, \quad (3.7)$$

which is Kähler with Kähler potential  $K = -\log g_{0\bar{0}}$ . The connections on  $\mathcal{L}$  and  $T\mathcal{M}$  are now given by  $\partial_i K$  and the metric connection  $\Gamma_{jk}^i$  for  $G_{i\bar{j}}$ . The covariant derivative of a section  $\xi \in \Gamma(\mathcal{L}^n \times T\mathcal{M}^m)$  is then given by



$$D_i \xi^{j_1 \dots j_m} = \partial_i \xi^{j_1 \dots j_m} + \Gamma_{ik}^{j_1} \xi^{k j_2 \dots j_m} + \dots + \Gamma_{ik}^{j_m} \xi^{j_1 \dots j_{m-1} k} + n \partial_i K \xi^{j_1 \dots j_m}. \quad (3.8)$$

In this picture, the Yukawa coupling is a symmetric rank 3 tensor with values in  $\mathcal{L}^2$ , which furthermore obeys the constraints

$$\partial_{\bar{l}} C_{ijk} = 0, \quad D_i C_{jkl} = D_j C_{ikl}. \quad (3.9)$$

Finally, for the curvature of the Zamolodchikov metric one obtains the relation

$$(R_{i\bar{j}})_l^k = [D_i, D_{\bar{j}}]_l^k = C_{ilm} C_{\bar{i}\bar{m}\bar{k}} e^{2K} G^{m\bar{m}} G^{k\bar{k}} - \delta_l^k G_{i\bar{j}} - \delta_i^k G_{l\bar{j}}. \quad (3.10)$$

The equations 3.7, 3.9 and 3.10 define the so called special Kähler geometry.

A Calabi-Yau threefold can be defined as Kähler manifold, which has a no-where vanishing  $(3,0)$  form  $\Omega(\underline{z})$ , depending on the complex structure deformations  $\underline{z}$ . We denote the mirror of  $M$  on which we evaluate the periods by  $W$ . One has simple formulas for the Kähler potential  $K$  and the Yukawa couplings  $C_{ijk}$  in terms of integrals over  $W$ . In particular

$$e^{-K} = i \int_W \Omega \wedge \bar{\Omega} =: (\Omega, \bar{\Omega}) \quad (3.11)$$

and

$$C_{ijk} = \int_W \Omega \wedge \partial_{z_i} \partial_{z_j} \partial_{z_k} \Omega. \quad (3.12)$$

One can reduce these integrals to period integrals and ultimately to certain solutions of the Picard-Fuchs equation as follows. First one chooses an integral symplectic basis  $\{A^k, B_k\}$ ,  $k = 1, \dots, h_{2,1}(W) + 1$  of three cycles in  $H_3(W, \mathbb{Z})$ , i.e.  $A^k \cap B_l = \delta_l^k$  such that all other intersections are zero, see [17]. Then one chooses a dual basis  $\{\alpha_l, \beta^k\}$ ,  $k = 1, \dots, h_{2,1}(W) + 1$  of three forms in  $H^3(W, \mathbb{Z})$ . It fulfills  $\int_{A^k} \alpha_l = \delta_l^k$ ,  $\int_{B_k} \beta^l = \delta_k^l$ , while all other pairings are zero. One has  $(\alpha_l, \beta^k) = i \delta_l^k$ , while again all other pairings are zero. Now we can expand

$$\Omega(z) = X^k(z) \alpha_k - F_l(z) \beta^l \quad (3.13)$$

in terms of the periods  $X^k(z) = \int_{A^k} \Omega(z)$  as well  $F_k(z) = \int_{B_k} \Omega(z)$ .

To recover the period integrals over the basis  $\{A^k, B_k\}$  from the solutions of the Picard-Fuchs equations we use special geometry and the typical degeneration of the periods at the point of maximal unipotent monodromy. First we note that the  $X^k$  serve as homogenous coordinates for the space of complex structures. As a consequence of Griffith transversality  $F^{(0)}(X^k) := \frac{1}{2} X^k F_k(X^k)$  is homogenous of degree 2 in  $X^k$  and  $F_k = \partial_{X^k} F^{(0)}$ .  $F^{(0)}$  is called the prepotential. At the point of maximal unipotent monodromy we have

$$\vec{\Pi} = \begin{pmatrix} \int_{B_1} \Omega \\ \int_{B_2} \Omega \\ \int_{A^1} \Omega \\ \int_{A^2} \Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ X_0 \\ X_1 \end{pmatrix} = \omega_0 \begin{pmatrix} 2\mathcal{F}^{(0)} - t \partial_t \mathcal{F}^{(0)} \\ \partial_t \mathcal{F}^{(0)} \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} \omega_3 + c \omega_1 + e \omega_0 \\ -\omega_2 - a \omega_1 + c \omega_0 \\ \omega_0 \\ \omega_1 \end{pmatrix}, \quad (3.14)$$

where  $\omega_0$  is the unique power series solution and  $\omega_k$  are solutions, which behave like  $\omega_0(z) \log(z)^k$  at infinity. The Frobenius method gives a canonical basis of these solutions.  $t = \frac{\omega_1}{\omega_0}$  is the mirror map and in terms of the latter the prepotential looks as follows

$$\mathcal{F}^{(0)} = -\frac{\kappa}{3!}t^3 - \frac{a}{2}t^2 + ct + \frac{e}{2} + f_{inst}(q), \quad (3.15)$$

where  $\kappa = H^3$ ,  $c = \frac{1}{24} \int_M c_2 \wedge H$ ,  $e = \frac{\zeta(3)\chi(M)}{(2\pi i)^3}$  and  $a = \frac{1}{2} \int_M i_* c_1(H) \wedge H$ . All these numbers are calculated on  $M$  using the formalism in section 2.1 and they fix the integral symplectic basis on  $W$  completely.

### 3.2 General solutions of the BCOV anomaly equation

The special geometry relations  $\bar{\partial}_i C_{jkl} = 0$  and  $D_i C_{jkl} = D_j C_{ikl}$  allow us to integrate the Yukawa coupling and its complex conjugate and express them through potential functions

$$C_{jkl} = D_j D_k D_l \mathcal{F}^{(0)}, \quad C_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}} D_{\bar{k}} D_{\bar{l}} \bar{\mathcal{F}}^{(0)}. \quad (3.16)$$

Here,  $\mathcal{F}_0^{(0)}$  is a  $C^\infty$  section of  $\mathcal{L}^2$  as  $C_{jkl}$  is such a section. Analogously,  $\bar{\mathcal{F}}^{(0)}$  is a  $C^\infty$  section of  $\bar{\mathcal{L}}^2$ . In the one moduli cases we are considering here equation 3.16 turns into

$$C_{zzz} = D_z D_z D_z \mathcal{F}^{(0)}(z, \bar{z}), \quad C_{\bar{z}\bar{z}\bar{z}} = D_{\bar{z}} D_{\bar{z}} D_{\bar{z}} \bar{\mathcal{F}}^{(0)}(z, \bar{z}). \quad (3.17)$$

The genus one free energy suffers from a holomorphic anomaly first calculated in [18],

$$\bar{\partial}_{\bar{k}} \partial_m \mathcal{F}^{(1)} = \frac{1}{2} \bar{C}_{\bar{k}}^{ij} C_{mij} - \left(\frac{\chi}{24} - 1\right) G_{\bar{k}m}. \quad (3.18)$$

This equation can be integrated straightforward and one obtains

$$\mathcal{F}^{(1)}(z) = \log(\det(G^{-1})^{\frac{1}{2}} e^{\frac{K}{2}(3+h^{2,1}-\frac{1}{12}\chi)} |f_1|^2), \quad (3.19)$$

where the holomorphic ambiguity is of the form

$$f_1(z) = \prod_i \Delta_i^{r_i} \prod_{i=1}^{h_{21}} z_i^{c_i}. \quad (3.20)$$

Here the  $\Delta_i$  are the components of the discriminant and the constants  $r_i$  and  $c_i$  are determined from the boundary behavior. In case of the conifold component of the discriminant  $\Delta_{con}$  the constant  $r_{con}$  is universally given by  $\frac{1}{12}$  as was first pointed out in [20]. The  $c_i$  are fixed by requiring the boundary condition

$$\lim_{z_i \rightarrow 0} \mathcal{F}^{(1)} = -\frac{1}{24} t_i \int_M c_2 \cdot H. \quad (3.21)$$

The higher genus generalization of the holomorphic anomaly is given through a recursion relation, the BCOV holomorphic anomaly equation ([9]),

$$\bar{\partial}_{\bar{k}} \mathcal{F}^{(g)} = \frac{1}{2} \bar{C}_{\bar{k}}^{ij} \left( D_i D_j \mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_j \mathcal{F}^{(g-1)} D_i \mathcal{F}^{(r)} \right), \quad (3.22)$$

where the  $\mathcal{F}^{(g)}$  are  $C^\infty$  sections of  $\mathcal{L}^{2-2g}$ .

The idea presented in [9] to solve this equation is to rewrite the right hand side as a derivative with respect to  $\bar{\partial}_{\bar{k}}$

$$\begin{aligned} \bar{\partial}_{\bar{k}} \mathcal{F}^{(g)} &= \bar{\partial}_{\bar{k}} \left( \frac{1}{2} S^{ij} \left( D_i D_j \mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_i \mathcal{F}^{(r)} D_j \mathcal{F}^{(g-1)} \right) \right) \\ &\quad - \frac{1}{2} S^{ij} \bar{\partial}_{\bar{k}} \left( D_i D_j \mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_i \mathcal{F}^{(r)} D_j \mathcal{F}^{(g-r)} \right), \end{aligned} \quad (3.23)$$

where  $S^{ij}$  is implicitly defined through

$$\bar{C}_{\bar{k}}^{ij} = \bar{\partial}_{\bar{k}} S^{ij}. \quad (3.24)$$

Using the commutator

$$R_{i\bar{k}j}^l = [D_i, \bar{\partial}_{\bar{k}}]_j^l = G_{i\bar{k}} \delta_j^l + G_{j\bar{k}} \delta_i^l - C_{ijm}^{(0)} \bar{C}_{\bar{k}}^{ml} \quad (3.25)$$

allows one to rewrite the second term in such a way that the  $\bar{\partial}_{\bar{k}}$  derivative acts directly on the  $\mathcal{F}^{(g)}$ . Then the holomorphic anomaly equations for  $g' < g$  can be used iteratively to generate an equation of the form

$$\bar{\partial}_{\bar{k}} \mathcal{F}^{(g)} = \bar{\partial}_{\bar{k}} \Gamma^{(g)}(S^{ij}, S^i, S, C_{i_1, \dots, i_n}^{(<g)}), \quad (3.26)$$

where  $S^i$ ,  $S$  and  $C_{i_1, \dots, i_n}^{(<g)}$  are defined through

$$\bar{C}_{j\bar{k}l} = e^{-2K} \bar{D}_{\bar{i}} \bar{D}_{\bar{j}} \bar{D}_{\bar{k}} S, \quad S_{\bar{i}} = \bar{\partial}_{\bar{i}} S, \quad S^j = G^{j\bar{k}} S_{\bar{k}}, \quad C_{i_1, \dots, i_n}^{(g)} = D_{i_1} \cdots D_{i_n} \mathcal{F}^{(g)}. \quad (3.27)$$

A solution is given by

$$\mathcal{F}^{(g)} = \Gamma^{(g)}(S^{ij}, S^i, S, C_{i_1, \dots, i_n}^{(<g)}) + f^{(g)}. \quad (3.28)$$

where  $f^{(g)}$  is the holomorphic ambiguity, which is not fixed by the recursive procedure. The method we will use to fix this ambiguity genus by genus is to go to boundary points of moduli space and use physical interpretation at those points to reconstruct the ambiguity globally. However, it is important to note that the boundary information is not restrictive enough to carry out the procedure up to genus infinity.

### 3.3 Topological limit and Gromov-Witten potentials

The topological limit of the free energy was introduced in [18]. In order to define it we first have to introduce the normalized solutions of the Picard-Fuchs equation around the large volume point in moduli space. As we are dealing with one parameter models, let these be given by  $\omega_0(z)$  and  $\omega_1(z)$ , which determines the mirror map to be  $t = t(z) = \frac{\omega_1(z)}{\omega_0(z)}$ . With these notations we can now introduce the topological limit to be defined by the following replacements,

$$G_{z\bar{z}} \rightarrow \frac{dt}{dz} \frac{d\bar{t}}{d\bar{z}}, \quad K_z \rightarrow -\partial_z \log \omega_0(z), \quad \mathcal{F}^{(g)}(z, \bar{z}) \rightarrow F^{(g)}(z), \quad (3.29)$$

in the solution 3.28, giving

$$F^{(g)}(z) = \Gamma(S^{zz}(z), S^z(z), S, C_r^{(<g)}(z)) + f_g(z). \quad (3.30)$$

This determines the  $F^{(g)}$  to be holomorphic prepotentials and sections of  $\mathcal{L}^{(2-2g)}$ . The Gromov-Witten potential is given through this holomorphic prepotential by

$$\begin{aligned} F_g(t) &= (\omega_0(z))^{2g-2} F^{(g)}(z) \\ &= (\omega_0(z))^{2g-2} \Gamma(S^{zz}, S^z, S, C_r^{(<g)}(z)) + (\omega_0(z))^{2g-2} f_g(z). \end{aligned} \quad (3.31)$$

This function is the generating function of the Gromov-Witten invariants  $N_g(d)$  and its expansion in terms of these is given by

$$F_g(t) = \frac{\chi}{2} (-1)^g \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!} + \sum_{d>0} N_g(d) q^d, \quad (q = e^{2\pi i t}), \quad (3.32)$$

where  $\chi$  is the Euler number of the Calabi-Yau manifold and  $B_g$  is the  $g$ th Bernoulli number.

For applications to the enumerative problem of counting holomorphic curves and/or the extraction of the physical content in terms of BPS states it is reasonable to switch to the effective action point of view. From this point of view the series  $F(\lambda, t) = \sum_{g=1}^{\infty} \lambda^{2g-2} F^{(g)}(t)$  computes the following term in the effective  $N = 2$  superpotential:

$$S_{1-loop}^{N=2} = \int d^4 x R_+^2 F(\lambda, t), \quad (3.33)$$

where  $R_+$  is the self-dual part of the curvature and  $\lambda$  is identified with the self-dual part of the graviphoton field strength  $F_+$ . Alternatively, this term is calculated by a one-loop integral in a constant graviphoton background, where the particles running in the loop are charged BPS states. The calculation is very similar to the ordinary Schwinger-loop calculation and the result is

$$\sum_{g \geq 0} \lambda^{2g-2} F_g(t) = \sum_{g \geq 0} \sum_{k \geq 1, d \geq 0} n_g(d) \frac{1}{k} (2 \sin \frac{k\lambda}{2})^{2g-2} q^{kd}. \quad (3.34)$$

The  $n_g(d)$  are the so-called Gopakumar-Vafa invariants and are integral.

## 4 The Grassmannian Calabi-Yau $(\mathbb{G}(2, 5)\|1, 1, 3)_{-150}^1$

This Calabi-Yau manifold is obtained as a complete intersection of hypersurfaces in the Grassmannian  $\mathbb{G}(2, 5)$  as described in section 2. In our special case the Plücker embedding is an embedding of  $\mathbb{G}(2, 5)$  into  $\mathbb{P}^9$  and equations 2.14 take the form

$$\begin{aligned} z_{23}z_{45} - z_{24}z_{35} + z_{25}z_{34} &= 0, \\ z_{13}z_{45} - z_{14}z_{35} + z_{15}z_{34} &= 0, \\ z_{12}z_{45} - z_{14}z_{35} + z_{15}z_{34} &= 0, \\ z_{12}z_{35} - z_{13}z_{25} + z_{15}z_{23} &= 0, \\ z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} &= 0. \end{aligned} \tag{4.1}$$

Now, the Calabi-Yau  $(\mathbb{G}(2, 5)\|1, 1, 3)_{-150}^1$  is defined to be a smooth 3-dimensional Calabi-Yau complete intersection of 3 hypersurfaces of degrees 1, 1 and 3 in  $\mathbb{P}^9$  with  $\mathbb{G}(2, 5)$ . A calculation shows that we have  $h^{1,1} = 1$ ,  $h^{2,1} = 76$  and  $\chi(M) = -150$ .

### 4.1 Picard-Fuchs differential equations and the structure of the moduli space

The Picard-Fuchs operator is given by:

$$\begin{aligned} \mathcal{P} = & -18z - 360z^2 + (-147z - 2106z^2)\theta + (-444z - 3969z^2)\theta^2 \\ & + (-594z - 2916z^2)\theta^3 + (1 - 297z - 729z^2)\theta^4, \end{aligned} \tag{4.2}$$

where  $\theta = z \frac{d}{dz}$ . As one can read off, the discriminant is given by  $\text{dis}(z) = 1 - 297z - 729z^2$ . The Yukawa coupling can be extracted from the Picard-Fuchs operator and its normalization is determined by the intersection number  $H^3$  given in section 2. This procedure is explained in [17] and the result for our particular example is

$$C_{zzz} = \frac{15}{z^3(1 - 11 \cdot 3^3 z - 3^9 z^2)}. \tag{4.3}$$

We expect the solutions to develop logarithmic singularities around the points  $\text{dis}(\alpha_i) = 0$ ,  $i \in \{1, 2\}$ , which indeed occur as can be seen from the index structure at these points:

$$(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 1, 1, 2). \tag{4.4}$$

These points are known as the conifold-points of the moduli space. As is known through the work of Strominger [19] at these points certain non-perturbative type II RR-states become massless and integrating them out leads to singularities in the Wilsonian effective action. Such a singularity occurs also in the free energies of the topological

string, as was first observed in [20], as these free energies calculate couplings of the four dimensional effective field theory. While calculating genus  $g$  topological string amplitudes we will make extensive use of the knowledge that such massless states exist to put restrictive bounds on the holomorphic ambiguity.

Another special point in our particular moduli space is the point at infinity. Here the Picard-Fuchs-operator develops the following indices:  $(\rho_1, \rho_2, \rho_3, \rho_4) = (\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3})$ . The  $\mathbb{Z}_3$ -symmetry at this point suggests that it is the enhanced symmetry point of a particular Landau-Ginzburg orbifold model. Putting regularity conditions on topological string free energies at this point gives us another bound on the holomorphic ambiguity and the resulting Gopakumar-Vafa invariants will give us a consistency check whether our regularity assumption was justified.

Finally, the structure of the singularities can be summarized in the following table

z	0	$\alpha_1$	$\alpha_2$	$\infty$
$\rho_1$	0	0	0	1/3
$\rho_2$	0	1	1	2/3
$\rho_3$	0	1	1	4/3
$\rho_4$	0	2	2	5/3

## 4.2 $g = 0$ and $g = 1$ Gopakumar-Vafa invariants

In this section we summarize the calculations of the genus zero and one Gopakumar-Vafa invariants for the Grassmannian. We will solve the Picard-Fuchs equation around the point  $z = 0$  and obtain the mirror map at this point.

The normalized regular solution and the linear-logarithmic solution are

$$\left. \begin{aligned} \omega_0(z) &= 1 + 18z + 1710z^2 + 246960z^3 + 43347150z^4 + \dots \\ \omega_1(z) &= \log x_0(z) + 75z + \frac{16497}{2}z^2 + 1257046z^3 + \frac{907324065}{4}z^4 + \dots \end{aligned} \right\} \quad (4.5)$$

The complexified Kähler modulus is defined through  $2\pi it = \frac{\omega_1(z)}{\omega_0(z)}$  and the  $q$ -expansion of the  $z$ -coordinate takes the following form:

$$z = q - 75q^2 + 1539q^3 - 60073q^4 + \dots, \quad (4.6)$$

where  $q := e^{2\pi it}$ .

Now, we are able to determine the quantum corrected Yukawa coupling  $K_{ttt}(t)$  at  $z = 0$ . It is given by

$$\left(\frac{1}{\omega_0(z)}\right)^2 C_{zzz} \left(\frac{dz}{dt}\right)^3 = 15 + 540q + 100980q^2 + 16776045q^3 + 2873237940q^4 + \dots \quad (4.7)$$

From these Yukawa couplings we can obtain the Gromov-Witten potential

$$K_{ttt}(t) = \left(q \frac{d}{dq}\right)^3 F_0(t). \quad (4.8)$$

The genus one invariants are obtained through the BCOV formula for the holomorphic potential which is the topological limit of 3.19

$$F^{(1)}(z) = \frac{1}{2} \log \left\{ \left( \frac{1}{\omega_0(z)} \right)^{3+h^{1,1}-\frac{\chi}{12}} \left( \frac{dz}{dt} \right) \text{dis}(z)^{-\frac{1}{6}} z^{c-1-\frac{c2.H}{12}} \right\}, \quad (4.9)$$

where we determine  $c = 0$  through the boundary behavior 3.21. As both zeros of the discriminant describe conifold points, it appears with factor  $-1/12$  in the logarithm.

Using the mirror map  $z = z(q)$  we finally obtain the genus one Gromov-Witten potentials

$$F_1^M(t) = F^{(1)}(z(q)). \quad (4.10)$$

### 4.3 Higher genus GV-Invariants

In this section we explain the recursive solution of the BCOV holomorphic anomaly equation found in [10] utilizing the polynomial structure of the partition functions. The topological limits at certain points in the moduli space are calculated giving boundary conditions on the holomorphic ambiguity.

#### 4.3.1 Recursive Solution of the BCOV equation

The general form 3.28 of the solution to 3.22 is not so useful for higher genus calculations as the procedure to determine the anholomorphic part grows exponentially with the genus. The situation can be improved once one notices that the terms appearing in the Feynman graph expansion are not completely independent, as was first observed in [7]. Using these interrelations, in [10] a recursive procedure for the quintic was developed whose complexity grows only polynomially with the genus.

The basic idea is to introduce two sets of generators, given by

$$A_k = G^{z\bar{z}} \theta_z^k G_{z\bar{z}}, \quad B_k = e^{K(z,\bar{z})} \theta_z^k e^{-K(z,\bar{z})}, \quad (4.11)$$

where  $\theta_z = z \frac{d}{dz}$ . A short calculation shows

$$\theta_z A_k = A_{k+1} - A_1 A_k, \quad \theta_z B_k = B_{k+1} - B_1 B_k. \quad (4.12)$$

Noticing the relation  $e^{-K(z,\bar{z})} = (\Omega(z), \bar{\Omega}(z))$ , the Picard-Fuchs equation corresponding to the Picard-Fuchs operator 4.2 can be rewritten in terms of the  $B_k$

$$B_4 = r_1(z) B_1 + r_2(z) B_2 + r_3(z) B_3 + r_4(z), \quad (4.13)$$

where the  $r_k(z)$  are rational functions.

Furthermore, there exists a similar relation for the  $A_k$ . As was shown in [10]  $A_2$  is given by

$$A_2 = -4B_2 - 2B_1(A_1 - B_1 - 1) + \theta_z \log(zC_{zzz})T_{zz} + r(z), \quad (4.14)$$

where  $T_{zz}$  is defined through the  $S^{zz}$  propagator

$$T_{zz} = -(zC_{zzz})S^{zz}, \quad (4.15)$$

and  $r(z)$  is a holomorphic function to be specified later. Also the propagators are defined up to holomorphic functions  $f$  and  $v$

$$\begin{aligned} S^{zz} &= \frac{1}{C_{zzz}} (2\partial \log(e^K |f|^2) - (G_{z\bar{z}}v)^{-1} \partial(vG_{z\bar{z}})) \\ &= -\frac{1}{zC_{zzz}} \left( 2B_1 + 2\frac{\partial f}{f} + A_1 - \frac{\partial v}{v} \right). \end{aligned}$$

We will make a choice of  $f$  and  $v$ , such that the invariant combinations  $e^K |f|^2$  and  $G_{z\bar{z}}|v|^2$  remain finite around  $z = 0$ . The calculation is most conveniently performed by taking the topological limit and we obtain  $v = z$  and  $f = 1$ . Therefore,  $T_{zz}$  takes the form

$$T_{zz} = 2B_1 + A_1 + 1. \quad (4.16)$$

The rational function  $r(z)$  is obtained by taking the topological limit of both sides of equation 4.14 and making the Ansatz

$$r(z) = c_0 + c_1 \frac{1}{dis(z)} + c_2 \frac{z}{dis(z)}. \quad (4.17)$$

The coefficients  $c_i$  are extracted by comparing both sides and we obtain

$$r(z) = -\frac{4}{9} + \frac{13}{9(1 - 297z - 729z^2)} - \frac{282z}{(1 - 297z - 729z^2)}. \quad (4.18)$$

The two equations 4.14 and 4.13 show that the  $\theta_z$ -derivative acts within the ring generated by  $A_1, B_1, B_2$  and  $B_3$ . More precisely, we have the property

$$\theta_z : \mathbb{C}(z)[A_1, B_1, B_2, B_3] \rightarrow \mathbb{C}(z)[A_1, B_1, B_2, B_3]. \quad (4.19)$$

Similarly, the action of the  $\partial_{\bar{z}}$  derivative just adds two more generators to the above polynomial ring, namely  $\partial_{\bar{z}}B_1$  and  $\partial_{\bar{z}}A_1$ . This is because, as was shown in [10] as well as in [14], one has the following identities

$$\partial_{\bar{z}}B_2 = (1 + A_1 + 2B_1)\partial_{\bar{z}}, \quad (4.20)$$



$$\partial_{\bar{z}} B_3 = (A_2 + 3B_1 + 3B_2 + 3A_1 B_1 + 1) \partial_{\bar{z}} B_1. \quad (4.21)$$

The next step will be to show that rewriting the holomorphic anomaly equations allows us to rewrite the solutions in terms of polynomials in  $A_1$ ,  $B_1$ ,  $B_2$  and  $B_3$ . In order to proceed we first introduce the quantities  $P_n^{(g)}$  defined through

$$P_n^{(g)} = (z^3 C_{zzz})^{g-1} z^n D_z^n \mathcal{F}^{(g)} \quad (n = 0, 1, 2, \dots). \quad (4.22)$$

Under the assumption that  $\partial_{\bar{z}} A_1$ ,  $\partial_{\bar{z}} B_1$  are independent the BCOV equation

$$\partial_{\bar{z}} P^{(g)} = \frac{1}{2} \partial_{\bar{z}} (z C_{zzz} S^{zz}) \left\{ P_2^{(g-1)} + \sum_{r=1}^{(g-1)} P_1^{g-1} P_1^{(r)} \right\} \quad (4.23)$$

can be translated into

$$\begin{aligned} 0 &= 2 \frac{\partial P^{(g)}}{\partial A_1} - \left( \frac{\partial P^{(g)}}{\partial B_1} + \frac{\partial_{\bar{z}} B_2}{\partial_{\bar{z}} B_1} \frac{\partial P^{(g)}}{\partial B_2} + \frac{\partial_{\bar{z}} B_3}{\partial_{\bar{z}} B_1} \frac{\partial P^{(g)}}{\partial B_3} \right), \\ \frac{\partial P^{(g)}}{\partial A_1} &= -\frac{1}{2} \left\{ P_2^{g-1} + \sum_{r=1}^{g-1} P_1^{(g-r)} P_1^{(r)} \right\}. \end{aligned}$$

This shows the polynomiality of the solutions. Performing the following variable change

$$\begin{aligned} u &= B_1, \quad v_1 = 1 + A_1 + 2B_1, \quad v_2 = -B_1 - A_1 B_1 - 2B_1^2 + B_2, \\ v_3 &= -B_1 - 2A_1 B_1 - 5B_1^2 - A_1 B_1^2 - 2B_1^3 + B_1 B_2 + B_3 \\ &\quad - B_1(r(z) + T_{zz} \theta_z \log(z C_z z z)), \end{aligned}$$

one can furthermore obtain  $\frac{\partial}{\partial u} P^{(g)} = 0$  which reduces the number of independent variables to three. Notice that the above equations are generic for all kinds of one parameter models, once  $r(z)$  is extracted from the truncation relation 4.14. The holomorphic anomaly equation can now be solved recursively with the initial data  $P_3^{(0)} = 1$  and  $P_1^{(1)}$ , given by

$$P_1^{(1)} = \frac{1}{2} \left\{ -A_1 - (2 + h^{11} - \frac{\chi}{12}) B_1 - 1 - \frac{c_2 \cdot H}{12} - \frac{\theta_z(dis(z))}{6 dis(z)} \right\}. \quad (4.24)$$

A nice way to perform the integration is given in [14].

However, the integration of the holomorphic anomaly still leaves us with the holomorphic ambiguity. The relation between the genus  $g$  free energy  $\mathcal{F}^{(g)}$ , the holomorphic ambiguity  $f_g(z)$  and the polynomials  $P^{(g)}$  is given by the following equation

$$\mathcal{F}^{(g)} = (z^3 C_z z z)^{(1-g)} P^{(g)} + f_g(z). \quad (4.25)$$

The Gromov-Witten potentials are once again obtained through equation 3.31, where one has to make the substitutions

$$A_1 \rightarrow \left(\frac{dz}{dt}\right)\theta_z\left(\frac{dt}{dz}\right), \quad B_k \rightarrow \frac{1}{\omega_0(z)}\theta_z^k\omega_0(z), \quad (4.26)$$

in the polynomial solutions  $\mathcal{F}^{(g)} = \mathcal{F}^{(g)}(A_1(z, \bar{z}), B_k(z, \bar{z}), z)$ .

## 4.4 Holomorphic ambiguity and boundary conditions

Requiring regularity of  $F_g(t)$  at  $z = 0$  and  $z = \infty$ , we parameterize the holomorphic ambiguity through the Ansatz

$$f_g(z) = a_0 + a_1z + \cdots + a_{2g-2}z^{2g-2} + \frac{c_0 + c_1z + \cdots + c_{4g-5}z^{4g-5}}{dis(z)^{2g-2}}. \quad (4.27)$$

From this we see that the total number of unknown parameters is  $6(g-1) + 1$  and grows linearly in  $g$ .

One of the main conceptual problems of topological string theory on compact Calabi-Yau is the determination of the holomorphic ambiguity. Boundary conditions may be given through the effective 4d action, but also, in some cases, geometrical considerations can be of use. For example, we can utilize the first few  $N_g(d)$  in the expansion of the Gromov-Witten potential once they are known through geometrical calculations. Usually, one puts the lower degree Gopakumar-Vafa invariants  $n_g(d)$  to zero as they count the number of genus  $g$  holomorphic curves in the Calabi-Yau. Once one knows that the  $n_g(d)$  are vanishing up a certain degree for a specific genus  $g$ , then one knows that they must be zero at least up to the same degree for genus  $g+1$ . This knowledge one can impose as boundary condition for the Gromov-Witten potentials. As boundary conditions from physical considerations are far more restrictive for higher genus calculations we will concentrate on these in this paper. In order to fix the ambiguity we evaluate the Gromov-Witten potentials at special points on the moduli space, where the physics is sufficiently well understood.

### 4.4.1 Expansion around the conifold points

Our model admits two conifold points and as was first observed in [11] each of them provides us with a gap-like structure in the higher genus topological string amplitudes which in turn impose  $2g-2$  conditions on the holomorphic ambiguity.

In order to make use of the gap condition we have to compute the topological limit around each conifold singularity. We denote the conifold singularity by  $c$ , i.e. in our case  $c$  stands for either  $\alpha_1 = 1/54(-11 - 5\sqrt{5})$  or  $\alpha_2 = 1/54(-11 + 5\sqrt{5})$ . In the following we will obtain a normalized set of solutions of the Picard-Fuchs differential equation. From the index structure around the conifold 5.4, the existence of a logarithmic solution can be deduced. Furthermore, we have solutions which start with  $s^i$  ( $s = (z - c)$ ,  $i = 0, 1, 2$ ) which we will denote by  $\omega_i^c(s)$ . We normalize the logarithmic solution  $\log(s)\omega_1^c(s) + \mathcal{O}(s^1)$  by requiring  $\omega_1^c(s) = s + \mathcal{O}(s^2)$ . The solution corresponding to the index  $\rho_4 = 2$  is

normalized to be of the form  $\omega_2^c(s) = s^2 + \mathcal{O}(s^3)$ . A suitable linear combination with  $\omega_1^c(s)$  and  $\omega_2^c(s)$  allows us to choose the solution for the index  $\rho_1 = 0$  to be of the form

$$\omega_0^c(s) = 1 + \mathcal{O}(s^3). \quad (4.28)$$

The mirror map can be now specified to be

$$k_t t_c = \frac{\omega_1^c(s)}{\omega_0^c(s)}, \quad (4.29)$$

where  $k_t$  is a constant which for the moment we can set to one.

We solve the Picard-Fuchs equations over the ring  $\mathbb{Q}[\alpha]/dis(\alpha)$  and obtain the following results for the periods and the mirror maps

$$\begin{aligned} \omega_0^\alpha(s) &= 1 + \frac{81}{250}(435709 + 1060776\alpha)s^3 + \mathcal{O}(s^4) \\ \omega_1^\alpha(s) &= s - \frac{3}{50}(3709 + 9126\alpha)s^2 + \frac{3}{25}(446957 + 1088046\alpha)s^3 + \mathcal{O}(s^4) \end{aligned} \quad (4.30)$$

$$s(t_\alpha) = t_\alpha - \frac{3}{50}(3709 + 9126\alpha)t_\alpha^2 + \frac{3}{50}(770597 + 1875852\alpha)t_\alpha^3 + \mathcal{O}(t_\alpha^4) \quad (4.31)$$

In order to regain the solutions around the points  $\alpha_i$ ,  $i \in \{1, 2\}$  one has to substitute  $\alpha$  by  $\alpha_i$ . For more details about this method see [14].

The topological limits around the conifold points are obtained by making the replacements

$$A_1(s + c, \bar{s} + \bar{c}) \rightarrow (s + c) \frac{d}{ds} \log \frac{dt_c}{ds}, \quad B_k \rightarrow \frac{1}{\omega_0^c(s)} \left( (s + c) \frac{d}{ds} \right)^k \omega_0^c(s) \quad (4.32)$$

in the defining relation 3.31.

The gap condition of [11] now tells us

$$F_c^{(g)}(t_c) = (\omega_0(s))^{2g-2} F_c^{(g)}(s) = \frac{\text{const.}}{t_c^{2g-2}} + \mathcal{O}(t_c^0), \quad (4.33)$$

for  $g \geq 2$ . This provides us with  $(2g - 2) - 1$  equations which are vanishing conditions for the coefficients of  $\frac{1}{t_c^i} (1 \leq i \leq 2g - 3)$ . Actually, the condition is even stronger as there exists a choice of the constant  $k_t$  under which in all higher genus expansions the leading term is of the form  $\frac{|B_{2g}|}{2g(2g-2)} \frac{1}{t_c^{2g-2}}$ .

It is interesting to have a look at this gap structure in the expansions of Gromov-Witten potentials once the holomorphic ambiguity is fixed completely,

$$\begin{aligned}
F_\alpha^{(2)}(t_\alpha) &= \frac{41 - 12276\alpha}{874800t_\alpha^2} + \frac{-14874743 + 3442099023\alpha}{36450000} + O(t_\alpha), \\
F_\alpha^{(3)}(t_\alpha) &= -\frac{5(-15005 + 4493016\alpha)}{4821232752t_\alpha^4} + \mathcal{O}(t_\alpha^0).
\end{aligned} \tag{4.34}$$

Again, substitute  $\alpha$  by  $\alpha_i$  to obtain the solutions around the specific vanishing point of the discriminant.

#### 4.4.2 Expansion around the orbifold point

The index structure 5.4 of the Picard-Fuchs operator suggests that the point at infinity is a  $\mathbb{Z}_3$  orbifold point. Therefore, we have to impose regularity of the free energies at this point in the moduli space. To obtain the topological limits we follow a path of argumentation presented in [14]. Let  $x$  be the coordinate at infinity, i.e.  $x = \frac{1}{z}$ . Then we can define  $\tilde{\mathcal{F}}^{(g)}(x, \bar{x})$  to be the solutions of the BCOV equation in  $x$ -coordinates with initial conditions  $\tilde{\mathcal{F}}_1^{(1)}(x, \bar{x})$  and  $\tilde{\mathcal{F}}_3^{(0)} = D_x D_x D_x \tilde{\mathcal{F}}^{(0)}(x, \bar{x})$ . On the other hand these initial conditions are related by

$$\tilde{\mathcal{F}}_3^{(0)}(x, \bar{x}) = C_{xxx}(x) = C_{zzz}\left(\frac{1}{x}\right)\left(\frac{dz}{dx}\right)^3 = \mathcal{F}_3^{(0)}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right)\left(\frac{dz}{dx}\right)^3. \tag{4.35}$$

From this we can infer that  $\tilde{\mathcal{F}}^{(g)}(x, \bar{x})$  and  $\mathcal{F}^{(g)}(z, \bar{z})$  are in the same coordinate patch of a trivialization of the line bundle  $\mathcal{L}$ , which again gives

$$\tilde{\mathcal{F}}^{(g)}(x, \bar{x}) = \mathcal{F}^{(g)}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right). \tag{4.36}$$

Therefore, the topological limit at infinity is simply obtained by setting  $\tilde{\mathcal{F}}^{(g)}(x, \bar{x}) = \mathcal{F}^{(g)}(A_1(\frac{1}{x}, \frac{1}{\bar{x}}), B_k(\frac{1}{x}, \frac{1}{\bar{x}}), \frac{1}{x})$  and taking the limits

$$A_1\left(\frac{1}{x}, \frac{1}{\bar{x}}\right) = \left(\frac{dz}{dx} \frac{d\bar{z}}{d\bar{x}} G^{x\bar{x}}\right)(-\theta_x) \left(\frac{dx}{dz} \frac{d\bar{x}}{d\bar{z}} G_{x\bar{x}}\right) \rightarrow -\left(\frac{dx}{dt_\infty}\right) \theta_x \left(\frac{dt_\infty}{dx}\right) - 2 \tag{4.37}$$

$$B_k\left(\frac{1}{x}, \frac{1}{\bar{x}}\right) = e^{\tilde{K}(x, \bar{x})} (-\theta_x)^k e^{-\tilde{K}(x, \bar{x})} \rightarrow \frac{1}{\omega_0^\infty(x)} (-\theta_x)^k \omega_0^\infty(x), \quad (k = 1, 2, 3), \tag{4.38}$$

where  $\omega_0^\infty(x)$ ,  $\omega_1^\infty(x)$  and  $t_\infty(x) = \frac{\omega_1^\infty(x)}{\omega_0^\infty(x)}$  are the periods and mirror map at infinity.

So in order to proceed we have to calculate these quantities first. From the index structure we have the following set of solutions,  $\omega_0^\infty(x) = x^{1/3} + \mathcal{O}(x^{4/3})$ ,  $\omega_1^\infty(x) = x^{2/3} + \mathcal{O}(x^{5/3})$ ,  $\omega_2^\infty(x) = x^{4/3} + \mathcal{O}(x^{7/3})$  and  $\omega_3^\infty(x) = x^{5/3} + \mathcal{O}(x^{8/3})$ . Using a linear combination with  $\omega_2^\infty(x)$  we can fix the first solution to be of the form

$$\omega_0^\infty(x) = x^{1/3} + \mathcal{O}(x^{7/3}). \tag{4.39}$$

Furthermore, the second solution can be fixed by taking a linear combination with the third solution to

$$\omega_1^\infty(x) = x^{2/3} + \mathcal{O}(x^{8/3}). \quad (4.40)$$

With these choices the relevant solutions are given by

$$\begin{aligned} \omega_0^\infty(x) &= x^{1/3} + \frac{x^{7/3}}{131220} - \frac{67}{51018336}x^{10/3} + \mathcal{O}(x^{13/3}), \\ \omega_1^\infty(x) &= x^{2/3} - \frac{2}{45927}x^{8/3} - \frac{467}{55801305}x^{11/3} + \mathcal{O}(x^{14/3}), \\ x &= t_\infty^3 - \frac{11}{102060}t_\infty^9 + \frac{12599}{595213920}t_\infty^{12} + \mathcal{O}(t_\infty^{15}). \end{aligned} \quad (4.41)$$

Using these data and the holomorphic limit discussed above we obtain the following Gromov-Witten potentials

$$\begin{aligned} F_\infty^{(2)}(t_\infty) &= \frac{\frac{41031}{160} + a_2}{t_\infty^4} + \frac{\frac{1367}{80} + a_1}{t_\infty} + \mathcal{O}(t_\infty), \\ F_\infty^{(3)}(t_\infty) &= \frac{\frac{22453281}{1600} + a_4}{t_\infty^8} + \frac{\frac{4572543}{3200} + a_3}{t_\infty^5} + \frac{-\frac{121464319}{567000} + a_2 + \frac{73a_4}{229635}}{t_\infty^2} + \mathcal{O}(t_\infty). \end{aligned} \quad (4.42)$$

As the orbifold point is a conformal field theory point and thus has to be regular, we see that demanding the vanishing of the coefficients of inverse powers of  $t_\infty$  gives us  $g$  conditions on the parameters of the holomorphic ambiguity.

Counting the number of boundary conditions from the orbifold and conifold points one notices that they are not yet enough to fix the ambiguity completely. This is no problem for lower genera as the vanishing of lower degree Gopakumar-Vafa invariants gives us enough conditions to fix all free parameters. On the other hand, as mentioned earlier, our example shows that there are not enough boundary conditions to solve the model up to genus infinity.

## 5 Other Models

We have analysed three other Calabi-Yau complete intersections in Grassmannians, namely  $(\mathbb{G}(2, 5)\|1, 2, 2)_{-120}^1$ ,  $(\mathbb{G}(3, 6)\|1^6)_{-96}^1$  and  $(\mathbb{G}(2, 6)\|1, 1, 1, 1, 2)_{-116}^1$ . All three admit interesting new features and share common properties with the model analysed previously. In particular, we have found a lense space point in the moduli space of the second model.

### 5.1 $(\mathbb{G}(2, 5)\|1, 2, 2)_{-120}^1$

The topological data of this Calabi-Yau are given by  $\chi = -120$ ,  $h^{2,1} = 61$ ,  $h^{1,1} = 1$ ,  $c_2 \cdot J = 68$ . The Picard-Fuchs operator which was obtained in [8] admits the following index structure

z	0	$\alpha_1$	$\alpha_2$	$\infty$
$\rho_1$	0	0	0	1/2
$\rho_2$	0	1	1	1/2
$\rho_3$	0	1	1	3/2
$\rho_4$	0	2	2	3/2

and the Yukawa coupling is determined to be

$$C_{zzz} = \frac{20}{z^3(1 - 11 \cdot 2^4 z - 2^8 z^2)}. \quad (5.1)$$

For the solutions around the conifold points we choose exactly the same normalization as in the case of  $(\mathbb{G}(2, 5)\|1, 1, 3)_{-150}^1$ . Looking at the point at infinity, we see that there are two logarithmic solutions. In order to obtain the mirror map only the first two solutions  $\omega_0^\infty$  and  $\omega_1^\infty$  are needed. They are of the form

$$\begin{aligned} \omega_0^\infty &= x^{1/2} + \mathcal{O}(x^{5/2}), \\ \omega_1^\infty &= \log(x)x^{1/2} + \mathcal{O}(x^{9/2}), \end{aligned} \quad (5.2)$$

and we take the mirror map to be of the form  $t = \frac{\omega_1^\infty(x)}{\omega_0^\infty(x)}$ .

With these conventions we calculate the expansions of the free energies around the singular points of the moduli space. We find the same gap conditions as in the case of  $(\mathbb{G}(2, 5)\|1, 1, 3)_{-150}^1$  around the two conifolds. The point at infinity turns out to be a regular point as we have to impose regularity on the Gromov-Witten potentials in order to obtain integral Gopakumar-Vafa numbers. We list the genus 2 and 3 expansions around this point

$$\begin{aligned}
F_\infty^2(t_\infty) &= \frac{5^{1/4}(136 + 3a_2)}{48\sqrt{3}t_\infty^{1/4}} + (a_1 + \frac{-119464 - 4047a_2}{32000}) + \mathcal{O}(t_\infty), \\
F_\infty^3(t_\infty) &= \frac{\sqrt{5}(\frac{1024}{3} + a_4)}{768\sqrt{t_\infty}} + \frac{-28849664 + 144000a_3 - 36423a_4}{460800\sqrt{3}5^{3/4}t_\infty^{1/4}} + \mathcal{O}(t_\infty). \quad (5.3)
\end{aligned}$$

As one can see regularity restrictions give us  $g - 1$  boundary conditions on the ambiguity.

## 5.2 $(\mathbb{G}(3, 6) \parallel 1^6)_{-96}^1$

This Calabi-Yau has the topological data  $\chi = -96$ ,  $h^{2,1} = 49$ ,  $h^{1,1} = 1$ ,  $c_2 \cdot J = 84$ . The Picard-Fuchs operator given in [8] admits the following index structure

z	0	$\alpha_1$	$\alpha_2$	$\infty$
$\rho_1$	0	0	0	4/3
$\rho_2$	0	1	1	1
$\rho_3$	0	1	1	1
$\rho_4$	0	2	2	5/4

The Yukawa coupling is given by

$$C_{zzz} = \frac{28}{z^3(1 - 26 \cdot 2^2 z - 27 \cdot 2^4 z^2)}. \quad (5.4)$$

The point at infinity admits one logarithmic solution which corresponds to a vanishing cycle and it appears that it also admits some orbifold features. The mirror map is given by  $t = \frac{\omega_1^\infty(x)}{\omega_0^\infty(x)}$ , where

$$\begin{aligned}
\omega_0^\infty &= x^{3/4} + \mathcal{O}(x^{7/4}), \\
\omega_1^\infty &= x + \mathcal{O}(x^2). \quad (5.5)
\end{aligned}$$

An interesting feature of this model is the fact that the two vanishing points of the discriminant, although having the same Picard-Fuchs-indices, behave differently when we analyze the Gromov-Witten potentials. In particular, the genus 1 Gromov-Witten potential of this model is

$$F^{(1)}(z) = \frac{1}{2} \log \left\{ \left( \frac{1}{\omega_0(z)} \right)^{3+h^{1,1}-\frac{\chi}{12}} \left( \frac{dz}{dt} \right) (-1+z)^{-\frac{1}{3}} (-1+64z)^{-\frac{1}{6}} z^{-1-\frac{c_2 \cdot H}{12}} \right\}. \quad (5.6)$$

This suggests that the point  $z = 1$  is not an ordinary conifold point but rather a lense space point, that is a point, where a cycle  $\mathcal{C}$  (for example  $S^3$ ) modded by a group  $G$  shrinks

to zero size. In the case of  $\mathcal{C} = S^3/G$  is a discrete subgroup of  $SU(2)$  and the resulting space  $S^3/G$  has fundamental group  $G$ . Spaces of this form were investigated in [21], where the number of BPS states admitted by such cycles was calculated. There it was argued that the number of D-brane bound states which are BPS is equal to the number of irreducible representations of  $G$  and their mass is given by the formula  $M_i = \mu d_i/G$  where  $\mu$  is the size of the unmodded cycle and  $d_i$  is the dimension of the  $i$ th irreducible representation of  $G$ . Comparing this with the genus one free energy of the topological string one finds

$$F^{(1)} = \sum_i -\frac{1}{12} \log(M_i) = \sum_i -\frac{1}{12} \log(\mu d_i/G). \quad (5.7)$$

In our particular example this is

$$F^{(1)} = -\frac{1}{12} \log(t_{1/64}) - \frac{2}{12} \log(t_1). \quad (5.8)$$

Using the identification  $t_1 = \mu/2$  we find from the above formula that the group  $G$  must be  $\mathbb{Z}_2$ . This also shows that two hypermultiplets are becoming massless at  $z = 1$ .

Our result is supported by the monodromy calculations made in [22]. There it was found that the monodromy matrix at the point  $z = 1$  is of Picard-Lefschetz form  $S_{\lambda,v}$ , where  $\lambda = 2$  which shows that this point is not an ordinary conifold point.

Higher genus calculations show that the ordinary gap condition holds at  $z = 1/64$  which is to be expected as this point is a conifold point. On the other hand the gap condition has to be slightly modified around  $z = 1$ . If we assume that the two hypermultiplets becoming massless are not interacting the modification to the leading term of the higher genus Gromov-Witten potential reads as follows

$$F_1^g(t_1) = 2 \frac{|B_{2g}|}{2g(2g-2)} \frac{1}{\mu^{2g-2}} + \mathcal{O}(t_1^0) = 2 \frac{|B_{2g}|}{2g(2g-2)} \frac{1}{2^{2g-2}} \frac{1}{t_1^{2g-2}} + \mathcal{O}(t_1^0). \quad (5.9)$$

This is exactly what we observe.

It remains to be discussed the point at infinity. It admits a gap-like structure as can be seen for example from the genus 4 expansion

$$F_\infty^4(t_\infty) = \frac{7}{240 t_\infty^6} + \frac{101797151}{11010048000} t_\infty^2 + \mathcal{O}(t_\infty^3). \quad (5.10)$$

### 5.3 $(\mathbb{G}(2,6) \parallel 1, 1, 1, 1, 2)_{-116}^1$

This manifold is characterized by the data  $\chi = -116$ ,  $h^{2,1} = 59$ ,  $h^{1,1} = 1$ ,  $c_2 \cdot J = 76$ . The structure of the solutions of the Picard-Fuchs operator is the following



z	0	$\alpha_1$	$\alpha_2$	$\infty$
$\rho_1$	0	0	0	1/2
$\rho_2$	0	1	1	2/3
$\rho_3$	0	1	1	4/3
$\rho_4$	0	2	2	3/2

The Yukawa coupling is given by

$$C_{zzz} = \frac{42}{z^3(1 - 65z - 64z^2)}. \quad (5.11)$$

The conifold locus is treated as usual. The mirror map at  $z = \infty$  is obtained by taking the ratio of the first two periods. They are of the form

$$\begin{aligned} \omega_0^\infty &= x^{1/2} + \mathcal{O}(x^{5/2}), \\ \omega_1^\infty &= x^{2/3} + \mathcal{O}(x^{5/3}). \end{aligned} \quad (5.12)$$

Now, our calculations show that the gap condition holds at the conifold locus. Furthermore, the point at infinity at first sight seems to be a regular orbifold point with  $\mathbb{Z}_6$ -symmetry and indeed this seems to be the case up to genus 3. But at genus 4 we find that the expansion of the Gromov-Witten potential around this point is singular. In particular we find

$$\begin{aligned} F_\infty^4(t_\infty) &= \frac{-\frac{8606402923}{164640} + a_6}{t_\infty^{18}} + \frac{-\frac{500305024099}{49787136} + a_5 - \frac{10}{63}a_6}{t_\infty^{12}} \\ &+ \frac{-\frac{443407050538901893}{179412923289600} + a_4 - \frac{20}{189}a_5 + \frac{831575}{54486432}a_6}{t_\infty^6} + \mathcal{O}(t_\infty^0), \end{aligned} \quad (5.13)$$

before fixing the ambiguity and

$$F_\infty^4(t_\infty) = \frac{2}{2187 t_\infty^6} + \frac{108172361}{131681894400} + \mathcal{O}(t_\infty), \quad (5.14)$$

after having fixed the ambiguity.

## 5.4 $(\mathbb{G}(2, 7) \parallel 1^7)_{-98}^1$

This manifold is characterized by the data  $\chi = -98$ ,  $h^{2,1} = 50$ ,  $h^{1,1} = 1$ ,  $c_2 \cdot J = 84$ . The structure of the solutions of the Picard-Fuchs operator is the following

z	0	$\alpha_1$	$\alpha_2$	$\alpha_3$	3	$\infty$
$\rho_1$	0	0	0	0	0	1
$\rho_2$	0	1	1	1	1	1
$\rho_3$	0	1	1	1	3	1
$\rho_4$	0	2	2	2	4	1

We see that the Picard-Fuchs differential operator has the property of maximally degeneration at both  $z = 0$  and  $z = \infty$ . It was found in [23] that the expansion about  $z = 0$  corresponds to the Kähler moduli of the Grassmannian Calabi-Yau  $M = (\mathbb{G}(2, 7) \parallel 1^7)_{-98}^1$ , and the expansion about  $z = \infty$  to that of a Pfaffian Calabi-Yau  $M'$ . In [14] the instanton calculations for this model were extended up to genus 5 and we confirm their results for low genus.

## 6 Conclusions

In this paper we analyzed the topological string on five one parameter Calabi-Yau spaces realized as complete intersections in Grassmannians. One result is that the gap condition at the conifold that was discovered in local geometries in [12] and global geometries in [11] is also present in the Grassmannian Calabi-Yau manifolds.

Since it involves subleading terms the gap condition is more than a local statement. The fact that leading behavior of the  $F_g(t_c)$  near the conifold point is given by the  $c = 1$  string is understood from the leading order local geometry of the nodal singularity [26, 25, 27] and is true in any choice of the local coordinate system which has the right scaling behavior of the complex structure parameterization. On the other hand the gap is sensitive to the global embedding, because it is only true in the flat coordinates for the complex structure parameters, whose form depends on global properties of the period integrals.

Unlike the toric one parameter Calabi-Yau the Grassmannian one parameter models have usually several conifolds at various values of  $z$  in their moduli space and all these have to fulfill the gap condition in order for the BPS invariants to be integer. In all cases we found explicitly integer BPS numbers for the symplectic invariants up to genus 5, which would be very interesting to confirm by methods of enumerative geometry.

We find that the model  $(\mathbb{G}(3, 6) \parallel 1^6)_{-96}^1$  has a conifold at  $z = \frac{1}{64}$  and a lense space  $S^3/\mathbb{Z}_2$  shrinking at  $z = 1$ . We find that at the lense space singularity the analysis of the leading terms is exactly as predicted in [21] and that in addition there is a full gap structure in the subleading terms. The physical interpretation is that the two BPS states do not interact and in particular do not form light bound states. This model has also at  $t_\infty$  a branch point of order 12, a single logarithmic solution and a full gap structure.

The models  $(\mathbb{G}(2, 5) \parallel 1, 1, 3)_{-150}^1$ ,  $(\mathbb{G}(2, 5) \parallel 1, 2, 2)_{-120}^1$  are regular at  $t_\infty = 0$  at least to genus 5. The first has regular solutions, which hints a CFT with an  $\mathbb{Z}_3$  automorphism at  $t_\infty = 0$ . In this model the BPS invariant  $n_6^4 = 5$  has been checked geometrically by Sheldon Katz, who found also the vanishing of the BPS invariants for the other model in accord with Castelnuovo's Theory.

The model  $(\mathbb{G}(2, 5) \parallel 1, 2, 2)_{-120}^1$  has two logarithmic solutions and a branch point order of 2. It is conceivable that higher  $F_g$  are not regular at  $t_\infty = 0$ .

The model  $(\mathbb{G}(2, 6) \parallel 1, 1, 1, 1, 2)_{-116}^1$  has two different conifolds with a full gap structure. At the point  $t_\infty = 0$  it has regular solutions with an  $\mathbb{Z}_6$  branching. Curiously we find that the integrality of the BPS require that it has singular behavior in the  $F_g$  for  $g > 3$ .

For the Rodland example  $(\mathbb{G}(2, 7) \parallel 1^7)_{-98}^1$ , which has two points of maximal unipotent monodromy we confirm the analysis of [14] for low genus.

Solving the topological string to all genus would be important to study black holes in five and four dimensions [24]. It is notable that the range of the topological data, which determine the semiclassical analysis of black holes take more extreme values for the Grassmannians than for the toric varieties. In particular  $c_2 \cdot H$  and the triple intersection  $H^3$  take the highest values for Grassmannian Calabi-Yau. This is very useful for comparing the semiclassical and the microscopic description of black holes along the lines of [24]. Indeed we find that the microscopic entropy the Richardson transforms converge within 4 % to the expected value of the macroscopic calculation. For reference we show one plot for the extreme value of  $H^3 = 42$  in Appendix C.

## Acknowledgments

We would like to thank Thomas Grimm, Sheldon Katz, Marcos Mariño, Min-xin Huang, Piotr Sulkowski, and Don Zagier for very valuable discussions.

## Appendix

## A Chern classes and topological invariants

$$\mathbb{G}(2, 5) \quad : \quad \int_{G(2,5)} \sigma_1^6 = 5, \quad \int_{G(2,5)} \sigma_2 \sigma_1^4 = 3, \quad \int_{G(2,5)} \sigma_3 \sigma_1^3 = 1,$$

$$\begin{aligned} (\mathbb{G}(2, 5) \| 1, 1, 3)_{-150}^1 & : \quad c((\mathbb{G}(2, 5) \| 1, 1, 3)_{-150}^1) \\ & = 1 + (5c_1(Q)^2 - c_2(Q)) \\ & \quad - (8c_1(Q)^3 + 5c_1(Q)c_2(Q) - 5c_3(Q)) + \cdots, \end{aligned}$$

$$\Rightarrow \chi = -150, \quad c_2 \cdot H = 66, \quad H^3 = 15.$$

$$\begin{aligned} (\mathbb{G}(2, 5) \| 1, 2, 2)_{-120}^1 & : \quad c((\mathbb{G}(2, 5) \| 1, 2, 2)_{-120}^1) \\ & = 1 + (4c_1(Q)^2 - c_2(Q)) \\ & \quad - (4c_1(Q)^3 + 5c_1(Q)c_2(Q) - 5c_3(Q)) + \cdots, \end{aligned}$$

$$\Rightarrow \chi = -120, \quad c_2 \cdot H = 68, \quad H^3 = 20.$$

$$\mathbb{G}(2, 6) \quad : \quad \int_{G(2,6)} \sigma_1^8 = 14, \quad \int_{G(2,6)} \sigma_2 \sigma_1^6 = 9, \quad \int_{G(2,6)} \sigma_3 \sigma_1^5 = 4,$$

$$\begin{aligned} (\mathbb{G}(2, 6) \| 1, 1, 1, 1, 2)_{-116}^1 & : \quad c((\mathbb{G}(2, 6) \| 1, 1, 1, 1, 2)_{-116}^1) \\ & = 1 + (4c_1(Q)^2 - 2c_2(Q)) \\ & \quad - (2c_1(Q)^3 + 6c_1(Q)c_2(Q) - 6c_3(Q)) + \cdots, \end{aligned}$$

$$\Rightarrow \chi = -116, \quad c_2 \cdot H = 76, \quad H^3 = 28.$$

$$\mathbb{G}(3, 6) \quad : \quad \int_{G(3,6)} \sigma_1^9 = 42, \quad \int_{G(3,6)} \sigma_2 \sigma_1^7 = 21, \quad \int_{G(3,6)} \sigma_3 \sigma_1^6 = 5,$$

$$\begin{aligned} (\mathbb{G}(3, 6) \| 1^6)_{-96}^1 & : \quad c((\mathbb{G}(3, 6) \| 1^6)_{-96}^1) \\ & = 1 + 2c_1(Q)^2 \\ & \quad - (6c_1(Q)c_2(Q) - 6c_3(Q)) + \cdots, \end{aligned}$$

$$\Rightarrow \chi = -96, \quad c_2 \cdot H = 84, \quad H^3 = 42.$$

$$\mathbb{G}(2, 7) \quad : \quad \int_{G(2,7)} \sigma_1^{10} = 42, \quad \int_{G(2,7)} \sigma_2 \sigma_1^8 = 28, \quad \int_{G(2,7)} \sigma_3 \sigma_1^7 = 14,$$

$$\begin{aligned} (\mathbb{G}(2, 7) \| 1^7)_{-98}^1 & : \quad c((\mathbb{G}(2, 7) \| 1^7)_{-98}^1) \\ & = 1 + (4c_1(Q)^2 - 3c_2(Q)) \\ & \quad - (7c_1(Q)c_2(Q) - 7c_3(Q)) + \cdots, \end{aligned}$$

$$\Rightarrow \chi = -98, \quad c_2 \cdot H = 84, \quad H^3 = 42.$$

## B Tables of Gopakumar-Vafa invariants

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	540	0	0	0	0	0
2	12555	0	0	0	0	0
3	621315	-1	0	0	0	0
4	44892765	13095	0	0	0	0
5	3995437590	17230617	-1080	0	0	0
6	406684089360	6648808835	921735	420	5	0
7	45426958360155	1831575868830	6512362740	-26460	-2160	0
8	5432556927598425	433375127634753	5837267557035	6528493485	218160	-2160
9	684486974574277695	94416986839804040	3061620003073095	20216637579465	6735865790	2770635
10	89872619976165978675	19571240651198871015	1223886411726167880	22818718255545315	85314971897190	5441786955

Table B.1: Gopakumar-Vafa invariants  $n_g(d)(g \leq 5)$  of the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(2, 5) \parallel 1, 1, 3)_{-150}^1$ .

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	400	0	0	0	0	0
2	5540	0	0	0	0	0
3	164400	0	0	0	0	0
4	7059880	1537	0	0	0	0
5	373030720	882496	0	0	0	0
6	22532353740	214941640	15140	0	0	0
7	1493352046000	37001766880	57840400	-800	0	0
8	105953648564840	5388182343297	36620960080	10792630	320	5
9	7919932042500000	715201587952800	12817600017680	33952864320	697600	-1600
10	616905355407694800	89732472170109248	3295335805457360	29386059424200	32052405340	-32320

Table B.2: Gopakumar-Vafa invariants  $n_g(d)(g \leq 5)$  of the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(2, 5) \parallel 1, 2, 2)_{-120}^1$ .

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	210	0	0	0	0	0
2	1176	0	0	0	0	0
3	13104	0	0	0	0	0
4	201936	0	0	0	0	0
5	3824016	84	0	0	0	0
6	82568136	74382	0	0	0	0
7	1954684008	8161452	0	0	0	0
8	49516091520	560512344	70896	0	0	0
9	1321186053432	31354814820	39198978	0	0	0
10	36729091812168	1568818990200	7239273552	1086246	0	0
11	1055613263065704	73339159104540	827701960638	932836632	1722	0
12	31184875579315920	3279169536538154	72679697259288	284870410986	55653752	0

Table B.3: Gopakumar-Vafa invariants  $n_g(d)(g \leq 5)$  of the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(3, 6)\|1^6)_{-96}^1$ .

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	280	0	0	0	0	0
2	2674	0	0	0	0	0
3	48272	0	0	0	0	0
4	1279040	27	0	0	0	0
5	41389992	26208	0	0	0	0
6	1531603276	5914124	-54	0	0	0
7	62153423432	745052912	56112	0	0	0
8	2699769672096	73219520613	120462612	-5267	0	0
9	123536738915800	6326648922384	40927354944	4713072	840	0
10	5890247824324990	506932941439940	8145450103430	15699104736	-91464	-404
11	290364442225572848	38717395881042032	1228133118935408	8307363701728	4174512664	66640
12	14713407331980050400	2863231551878100494	156147718274297768	2460694451990694	7534787308968	991403118

Table B.4: Gopakumar-Vafa invariants  $n_g(d)(g \leq 5)$  of the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(2, 6) \parallel 1, 1, 1, 1, 2)_{-116}^1$ .

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
1	196	0	0	0	0	0
2	1225	0	0	0	0	0
3	12740	0	0	0	0	0
4	198058	0	0	0	0	0
5	3716944	588	0	0	0	0
6	79823205	99960	0	0	0	0
7	1877972628	8964372	0	0	0	0
8	47288943912	577298253	99960	0	0	0
9	1254186001124	31299964612	47151720	-1176	0	0
10	34657942457488	1535808070650	7906245550	325409	0	0
11	990133717028596	70785403788680	858740761340	956485684	-25480	3675
12	29075817464070412	3129139504135680	73056658523632	301227323110	27885116	73892

Table B.5: Gopakumar-Vafa invariants  $n_g(d)(g \leq 5)$  of the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(2, 7)\|1^7)_{-98}^1$ .



d	$g = 0$	$g = 1$	$g = 2$	$g = 3$
1	588	0	0	0
2	12103	0	0	0
3	583884	196	0	0
4	41359136	99960	0	0
5	3609394096	34149668	12740	0
6	360339083307	9220666238	25275866	1225
7	39487258327356	2163937552736	21087112172	22409856
8	4633258198646014	466455116030169	11246111235996	58503447590
9	572819822939575596	95353089205907736	4601004859770928	67779027822044
10	73802503401477453288	18829753458134112872	1586777390750641117	50069281882780727

d	$g = 4$	$g = 5$
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	25371416	3675
9	216888021056	33575388
10	521484626374894	1111788286385

Table B.6: Gopakumar-Vafa invariants  $n_g(d)$  ( $g \leq 5$ ) of the Pfaffian Calabi-Yau threefold  $M'$ .

## C 5D Blackhole asymptotic

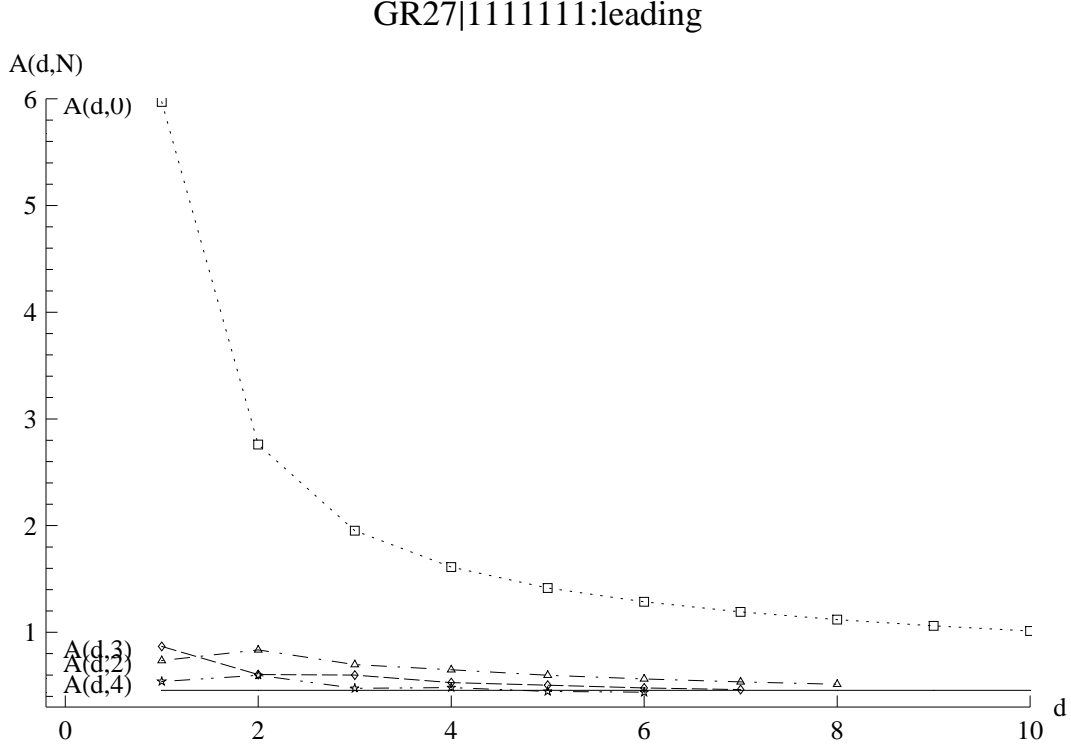


Figure 1: Leading behavior of the microscopic entropy for the 5d black hole for the Grassmannian Calabi-Yau threefold  $(\mathbb{G}(2, 7) \parallel 1, 1, 1, 1, 1, 1, 1)_1^1_{98}$ .  $A(d, m)$  are the Richardson transforms. The Richardson transforms of the microscopic entropy converge within 4 % to the expected value from the macroscopic calculation  $b_0 = \frac{4\pi}{3\sqrt{2}H^3} \sim .046$  for  $H^3 = 42$ , see [24] for details.

## References

- [1] A. Borel and F. Hirzebruch, “Characteristic classes and homogeneous spaces I”, Amer. J. Math. 80 1958 458-538.
- [2] E. Witten, “Phases of N=2 theories in two dimensions”, Nucl. Phys. **B403** (1993) 159 [hep-th/9301042].
- [3] M. Kontsevich, “Enumeration of rational curves via Torus actions”, [arXiv:hep-th/9405035].
- [4] A. B. Givental, “Elliptic Gromov - Witten invariants and the generalized mirror conjecture,” Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107, World Sci. Publishing, River Edge, NJ, 1998, [math.AG/9803053] .

- [5] A. Zinger, “Standard vs. Reduced Genus-One Gromov-Witten Invariants”, [arXiv:07060715] and “The Reduced Genus-One Gromov-Witten Invariants of Calabi-Yau Hypersurfaces”, [arXiv:07052397].
- [6] R. Gopakumar and C. Vafa, “M-theory and topological strings I”, [arXiv:hep-th/9809187] and “M-theory and topological strings II”, [arXiv:hep-th/9812127].
- [7] S. Katz, A. Klemm and C. Vafa, “M-theory, topological strings and spinning black holes”, Adv. Theor. Math. Phys. 3, 1445 (1999) [arXiv:hep-th/9910181].
- [8] V. Batyrev, I. Ciocan-Fontanine, B. Kim and D. v. Straten, “Conifold Transitions and Mirror Symmetry for Calabi-Yau Complete Intersections in Grassmannians,” alg-geom/9710022.
- [9] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes”, [arXiv:hep-th/9309140].
- [10] S. Yamaguchi, S. T. Yau, “Topological String Partition Functions as Polynomials”, [arXiv:hep-th/0406078].
- [11] M. Huang, A. Klemm and S. Quackenbush, “Topological string theory on compact Calabi-Yau: modularity and boundary conditions”, [arXiv:hep-th/0612125].
- [12] M. x. Huang and A. Klemm, “Holomorphic anomaly in gauge theories and matrix models”, JHEP **0709** (2007) 054, [arXiv:hep-th/0605195].
- [13] T. W. Grimm, A. Klemm, M. Marino and M. Weiss, “Direct integration of the topological string”, JHEP **0708** (2007) 058 [arXiv:hep-th/0702187].
- [14] S. Hosono and Y. Konishi, “Higher genus Gromov-Witten invariants of the Grassmannian, and the Pfaffian Calabi-Yau threefolds”, [arXiv:math.AG/0704.2928].
- [15] P. Griffiths and J. Harris, “Principles of Algebraic Geometry”.
- [16] B. Sturmfels, “Gröbner Bases and Convex Polytopes”, Univ. Lect. Notes, vo. 8, AMS, 1996.
- [17] P. Candelas, X.C. de la Ossa, P.S. Green, and L. Parkes, “A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory”, Nucl. Phys. B356 (1991), 21-74.
- [18] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic Anomalies in Topological Field Theories”, [arXiv:hep-th/9302103].
- [19] A. Strominger, “Massless Black Holes and Conifolds in String Theory”, [arXiv:hep-th/9504090].
- [20] C. Vafa, “A Stringy test of the fate of the conifold”, Nucl. Phys. B 447, 252 (1995) [arXiv:hep-th/9505023].

- [21] R. Gopakumar and C. Vafa, “Branes and Fundamental Groups”, [arXiv:hep-th/9712048].
- [22] C. van Enckevort and D. van Straten, “Monodromy calculations of fourth order equations of Calabi-Yau type”, [arXiv:math.AG/0412539].
- [23] E.A. Rodland, “The Pfaffian Calabi-Yau, its Mirror and their link to the Grassmannian  $G(2, 7)$ ”, *Compositio Math.* **122** (2000), no. 2, 135 - 149, [arXiv:math.AG/9801092].
- [24] M. x. Huang, A. Klemm, M. Marino and A. Tavanfar, “Black Holes and Large Order Quantum Geometry,” arXiv:0704.2440 [hep-th].
- [25] R. Dijkgraaf and C. Vafa, arXiv:hep-th/0302011.
- [26] D. Ghoshal and C. Vafa, *Nucl. Phys. B* **453**, 121 (1995) [arXiv:hep-th/9506122].
- [27] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Commun. Math. Phys.* **261** (2006) 451 [arXiv:hep-th/0312085].